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Canonical Quantization of 1 + 1 Dimensional Gravity[1]

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Based on Polyakov's evaluation of the Fadeev–Popov determinant for (1 + 1)-dimensional gravity in the conformal gauge we formulate a canonical quantization in the synchronous gauge. We find that the system is describable as a quantum mechanical system of one degree of freedom. The quantization can be carried out and solved when any number of gauge fields are included. Scalar and spinor fields lead to new difficulties. For positive cosmological constant the geometry collapses as suggested by the classical system. For negative cosmological constant a more interesting behavior involving exponentially expanding and contracting universes occurs.

1. INTRODUCTION

The study of low-dimensional systems has proven to be a fruitful exercise in many branches of physics and has been particularly useful in helping to unravel the dynamics of non-Abelian gauge fields. It is natural to try to employ this strategy in attempting to understand the connection between quantum mechanics and Einstein's geometrical theory of gravitation. Surely many people have had this idea many times and just as surely they have been put off by the statement that classical general relativity is trivial in space-time dimensions lower than 4. More precisely, in 2 + 1 dimensions, the only solutions of Einstein's equations with cosmological constant are space-times of constant curvature, while in 1 + 1 dimensions $\int\sqrt{g}\,R$ is a topological invariant and the only solution is $g_{\mu\nu} = 0$. There do not seem to be enough degrees of freedom to support a quantum field

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theory. In fact in $1+1$ dimensions there are not, but we will see that this objection does not prevent us from constructing a reasonable quantum theory.

$1+1$ Dimensions seems to be particularly annoying. The Lagrangian $\mathscr{L} = -\lambda_0\sqrt{(-g)}^{1/2}$ contains no derivatives and we cannot define canonical momenta, let alone a Hamiltonian. Light is thrown on the problem by the following fact, which we abstract from Polyakov's beautiful attempt to construct a viable string model (Polyakov, 1981):

Consider the path integral

$$Z = \int dg_{\mu\nu} \exp i \int (-g)^{1/2} (K_0 R - \lambda_0) \tag{1}$$

which, according to the work of Feynman, DeWitt, Mandelstam, Fadeev, and Popov (1967), is an appropriate starting point for constructing a quantum theory of gravity. As in any other gauge theory the path integral (1) is ill defined until we choose a gauge and compute the Fadeev–Popov determinant. Polyakov chooses the conformal gauge

$$g_{\mu\nu} = (\exp 2\phi)\eta_{\mu\nu} \tag{2}$$

and computes the determinant in closed form:

$$Z = \int d\phi \exp i = \left[\frac{26}{48\pi} \int (\partial_\mu \phi)^2 + \tilde{\lambda} e^{2\phi} + \tilde{\kappa} \nabla^2 \phi \right] \tag{3}$$

where $\tilde{\kappa}$ (which we drop since it is a total divergence) and $\tilde{\lambda}$ are (infinitely) renormalized constants.

We are led to the surprising conclusion that $(1+1)$-dimensional quantum gravity is in some sense equivalent to the theory of a single scalar field. More importantly, perhaps, the theory has derivatives in it and thus has a chance of having interesting dynamics. The question now is, in precisely what sense is gravity equivalent to the Liouville field theory? Does it really have so many degrees of freedom?

There is an important technical problem with the quantization of (3) which relates directly to these questions. The classical Liouville Lagrangian is invariant under the conformal transformations,

$$\varphi(x+t, x-t) \to \varphi(f(x+t), g(x-t)) + \ln f'(x+t) g'(x-t)$$

for any analytic f and g. This is a remnant of the general coordinate invariance of the original Lagrangian. Like all such residual gauge symmetries it must be preserved by any quantization of $L_{\text{Liouville}}$ that purports to be quantum gravity. Neither the standard canonical quantization[4] (Jackiw and 'Hoker, 1982), nor more sophisticated methods (Thorn et al., 1982; Gervais and Neveu, 1983) based on exact classical solutions, preserve these symmetries.

Assuming that a conformally invariant quantization procedure can be found the residual coordinate invariance will require the imposition of constraints in the Hilbert space of the Liouville field theory. Physical states must be annihilated by the generators of all residual gauge (conformal) transformations. This may drastically reduce the number of true degrees of freedom in the problem.

In conclusion we see that although Polyakov's work raises the hope of finding a nontrivial quantum dynamics for gravity in $1+1$ dimensions it leaves many questions unanswered. In the next two sections we will try to resolve these puzzles by looking at $(1+1)$-dimensional gravity from a completely different point of view.

2. THE SYNCHRONOUS GAUGE

Our basic intuitions about $(1+1)$-dimensional gravity are based on J. A. Wheeler's "superspace" formulation of the dynamics of gravitation. According to Wheeler (Misner et al., 1970), the basic dynamical variable of $(d+1)$-dimensional general relativity is a d geometry. The quantum states of the system are represented by functions on the space of all d geometries —superspace.

For definiteness we shall make some topological restrictions on superspace. Space is assumed connected at all times. In one space dimension only two choices are possible; open or closed periodic geometries. Whenever a choice must be made we will look only at a closed universe.

For $d=1$ a smooth closed connected d manifold is completely characterized by a single parameter—its volume v. Thus in $1+1$ dimensions quantum gravity should be a simple quantum mechanics problem with one degree of freedom.

In classical general relativity the superspace idea can be formulated without choosing a gauge (Arnowitt et al., 1962). The Einstein equations

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 0 \tag{4}$$

can be derived as the variational equations of the action $\int \sqrt{g}\ R$. The $d+1$ metric $g_{\mu\nu}$ is varied over all values which interpolate between a given pair of d geometries. The space-space components of the equations are dynamical, while the space-time and time-time equations are constraint equations analogous to Gauss' law in gauge theories. They express the vanishing of the generators of the local symmetries of the theory. The space-time equation is the generator of reparametrizations of the initial d-dimensional surface, while the time-time equation is the generator of local time translations, i.e., the Hamiltonian density. Thus we obtain the Wheeler–DeWitt (W–D) equations

$$\mathscr{H}(x) = 0 \tag{5}$$

$$\mathbf{P}(x) = 0 \tag{6}$$

The time-time and space-time components of the metric [the lapse and shift functions (Arnowitt et al., 19)] are not determined by the equations (they are analogous to A_0 in gauge theories) but are chosen to simplify the particular solution under study. (Wheeler's picturesque name for this is "many fingered time.") This convenient classical procedure cannot be applied in quantum mechanics. In order to formulate the theory in a Hilbert space we must choose a gauge, i.e., fix the value of the lapse and shift functions. In the quantum theory the equations that follow from variation of g_{00} and g_{0i} will be imposed as conditions on physical states in the Hilbert space. It should be emphasized that this procedure is independent of the particular form of the Einstein action. The absence of equations for g_{00} and g_{0i}, and the necessity of imposing constraints on the states is a consequence of general covariance. Different generally covariant actions will lead to different forms for the generators, but the Wheeler–DeWitt equations will always be valid.

Most previous discussions of the canonical quantization of gravity (Teitelboim, 1983) have used the synchronous or proper-time gauge (s gauge)

$$\begin{aligned} g_{00} &= 1 \\ g_{0i} &= 0 \end{aligned} \tag{7}$$

We shall follow this tradition, but we find it more convenient to work with zweibeins e_μ^α instead of metric components. Thus we impose the constraints

$$\begin{aligned} e_0^0 &= 1 \\ e_0^1 &= 0 = e_1^0 \\ e_1^1 &\equiv e \end{aligned} \tag{8}$$

which can be achieved by combining a local Lorentz transformation with a general coordinate transformation. One first picks a spacelike surface Γ in the two geometry and coordinatizes it in an arbitrary way. The local time axes are then chosen to be the geodesics which intersect Γ perpendicularly and coordinate time is set equal to proper time. This fixes $g_{00} = 1$ and $g_{0i} = 0$. Finally a local Lorentz transformation is performed to align the zweibein with the coordinate axes.

In general, the coordinates defined in this way are not globally regular. Even if space-time is flat, the synchronous coordinates defined off a curved initial surface will be singular, as illustrated in Figure 1. The multiple coordinate labeling of a single point which is evident in this figure is a nuisance. More disturbing is the fact that the surfaces of constant time are not smooth 1-manifolds even for completely smooth 2-geometries (see Figure 2). These facts raise serious questions about the utility of the synchronous gauge.

We believe that these problems are not as serious as they seem though we do not have a rigorous argument that this is so. Such an argument would go something like this: The object we would like to construct is the amplitude for a given spatial geometry to evolve into another one. In the synchronous gauge one can ask the more detailed question: What is the amplitude for this transition in a given proper time τ? But general covariance [and in particular the Wheeler–DeWitt equation $H(x) = 0$] tell us that this more detailed information is illusory, the amplitude is indepen-

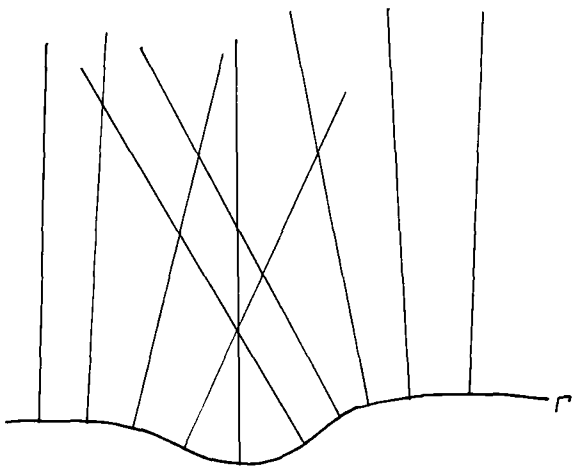

Fig. 1. Erection of a synchronous coordinate system.

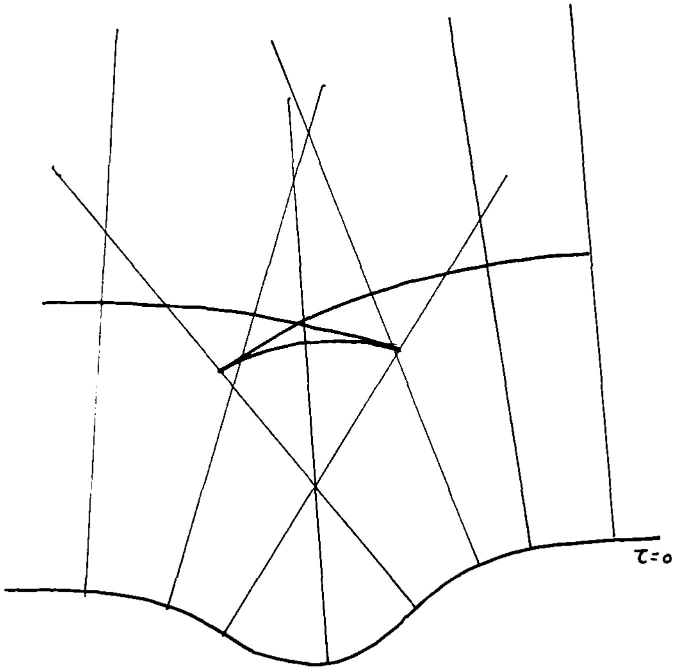

Fig. 2. Pathology of synchronous coordinates.

dent of the proper time. Thus without loss of generality we can consider a transition which takes place in an infinitesimal proper-time $d\tau$. For any smooth 2-manifold which connects smooth 1-manifolds over an infinitesimal proper time interval the synchronous coordinates are nonsingular. The pathologies we have described above can only occur for manifolds which are kinky on arbitrarily short distance scales. We believe that this is an ultraviolet problem which will manifest itself only as a renormalization of parameters in the Hamiltonian which describes the dynamics of smooth manifolds. At any rate, we will ignore the problem from here on.

The synchronous gauge conditions (8) are invariant under two classes of spatially local coordinate transformations. These are reparameterizations of the initial surface, under which e transforms as

$$e(x,t) \to e(x,t) + \partial_x[f_1(x)e] \tag{9}$$

and changes in the position of the surface, under which

$$e \to e + f^0(x)\partial_t e(x,t) - \partial_x\left[\int_0^t \frac{ds}{e^2(x,s)}\partial_x f_0(x)e(xt)\right] \tag{10}$$

The generator of spatial translations is

$$P(z) = e(z)\partial_z \Pi(z) \tag{11}$$

where

$$\Pi(z) = \frac{1}{i}\frac{\delta}{\delta e(z)} \tag{12}$$

and the equation $P(z)|\psi\rangle = 0$ is solved by

$$\psi(e(x)) = \psi\left(\int dx |e(x)|\right) = \psi(v) \tag{13}$$

This implies, as we had anticipated, that quantum gravity reduces to a problem with one degree of freedom.

It remains to find the Hamiltonian density $\mathscr{H}(x)$ in the synchronous gauge and to solve the Wheeler–DeWitt equation

$$\mathscr{H}(x)\psi(v) = 0 \tag{14}$$

In principle this could be done by starting from the path integral

$$\int de_\mu^\alpha \exp\left(-i\lambda \int d^2x \,|\det e|\right) \tag{15}$$

where e_μ^α ranges over all zweibeins with a given pair of 1-geometries as boundaries. This involves computing the Fadeev–Popov determinant in the s gauge which we have not done. Instead, in the next section we present a (Teitelboim, 1983) semiclassical determination of $\mathscr{H}(x)$ based on Polyakov (1981).

Before turning to this semiclassical analysis we want to introduce another gauge which is related to the synchronous gauge as the axial gauge is related to the $A_0 = 0$ gauge. At any fixed time t we can perform a spatial coordinate transformation to make $e(x, t)$ constant. The axial gauge is defined by making a *time-dependent* coordinate transformation so that

$$\begin{aligned} e_1^1(x,t) &= v(t), \qquad & e_0^0(x,t) &= 1 \\ e_1^0(x,t) &= 0, \qquad & e_0^1(xt) &\equiv y(x,t) \end{aligned} \tag{16}$$

We will find this gauge useful in discussing the coupling of gravity to matter.

3. THE HAMILTONIAN

In the conformal gauge (c gauge) defined by (2) the Fadeev–Popov determinant provides an effective action given by (Polyakov, 1981)

$$\beta \int \frac{(\partial_\mu \phi)^2}{2} - \lambda e^{2\phi}$$

$$\beta \equiv \frac{26}{24\pi} \tag{17}$$

where λ is the sum of the bare cosmological constant and a quadratically divergent correction. It can be chosen to vanish or to be a finite number of either sign.

We shall begin by treating (17) as a classical Lagrangian and transforming the equations into the s gauge. The equation of motion

$$\Box\phi + 2\lambda e^{2\phi} \tag{18}$$

is known as the Liouville equation. In the c gauge the curvature invariant R and the determinant of the metric are given by

$$R = e^{-2\phi}\Box\phi \tag{19}$$

$$|-g| = e^{2\phi} \tag{20}$$

Thus equation (18) may be written[3]

$$R(x,t) = -2\lambda \tag{21}$$

so that the curvature scalar is constant.

Now consider a spacelike curve Γ which we will use to erect a synchronous gauge. The surface Γ is identified as the surface $\tau = 0$. It may be chosen to satisfy a further gauge fixing constraint. We choose it so that the quantity $\mathscr{H}(x)$

$$\mathscr{H}(x) = \beta\left(\frac{\dot{e}^2}{2e} + \lambda e\right) \tag{22}$$

vanishes on Γ. The dot in (22) denotes differentiation with respect to the

[3] C. Teiteboim and R. Jackiw (1981) have independently suggested that the covariant equation $R = -2\lambda$ is the appropriate analog of Einstein's field equations in $1+1$ dimensions. From this point on the reader may interpret our analysis as a quantization of this equation in the temporal gauge.

proper time τ. It is easy to show that this condition is invariant under reparametrization of Γ.

Let us now prove that

$$\mathscr{H}(x,\tau)=0 \tag{23}$$

is satisfied for all time. To prove this we note that in the s-gauge the curvature R is given by

$$R=\ddot{e}/e \tag{24}$$

Therefore (21) implies

$$\frac{\ddot{e}}{e}=-2\lambda \tag{25}$$

Now consider the time derivative of $\mathscr{H}(x)$

$$\begin{aligned}\dot{\mathscr{H}}&=\beta\left[\frac{\dot{e}\ddot{e}}{e}-\frac{\dot{e}^3}{2e^2}+\lambda\dot{e}\right]\\&=\beta\dot{e}\left[\frac{\ddot{e}}{e}-\frac{\dot{e}^2}{2e^2}+\lambda\right]\end{aligned} \tag{26}$$

Using (25) gives

$$\dot{\mathscr{H}}=-\frac{\dot{e}}{e}\mathscr{H} \tag{27}$$

Thus if $\mathscr{H}$ is initially zero it remains so for all τ.

We can derive the equation of motion (25) and equation (23) from a Lagrangian and the W–D equations. Choose

$$L=\int\beta\left(\frac{\dot{e}^2}{2e}-\lambda e\right)dx \tag{28}$$

(Note that L is reparametrization invariant). The Hamiltonian density corresponding to (28) is just $\mathscr{H}(x)$,

$$\mathscr{H}(x)=\beta\left(\frac{\dot{e}^2}{2e}+\lambda e\right)=\frac{e\Pi_e^2}{2\beta}+\beta\lambda e \tag{29}$$

where Π_e is canonically conjugate to e. The Euler–Lagrange equation following from (28) is

$$\frac{d}{dt}\frac{\dot{e}}{e} = -\frac{\dot{e}^2}{2e^2} - \lambda$$

or

$$\frac{\ddot{e}}{e} = \left(\frac{\dot{e}^2}{2e^2} + \lambda\right) - 2\lambda \tag{30}$$

Now using the W–D equation (23) just gives (25).

The vanishing of the generators of local spatial translations also follows. The local generator is

$$P(x) = e\frac{\partial \Pi}{\partial x} \tag{31}$$

Using (23) we find

$$\Pi = \beta\frac{\dot{e}}{e} = -(-2\lambda\beta)^{1/2} \tag{32}$$

$$\partial_x \Pi = 0 \tag{33}$$

From which it follows that

$$P(x) = 0 \tag{34}$$

Let us now solve the equations of motion. From (25) we find

$$e = a(x)e^{i(2\lambda)^{1/2}t} + b(x)e^{-i(2\lambda)^{1/2}t} \tag{35}$$

The two cases $\lambda > 0$ and $\lambda < 0$ must be treated separately. For $\lambda > 0$ the condition that e be real requires $a = b$. Then requiring $\mathscr{H} = 0$ fixes $a = b = 0$ so that classically only a collapsed space is possible. For $\lambda < 0$ the situation is more interesting. In this case a need not equal b. In fact $\mathscr{H} = 0$ requires either $a = 0$ or $b = 0$. The two cases are related by time reversal and describe exponentially contracting or expanding universes. (Note that the relation between the sign of λ and the behavior of the space-time geometry is opposite to the four-dimensional case.)

If the spatial geometry is closed then x is a periodic variable with period 1. In this case the expanding geometry looks like Figure 3.

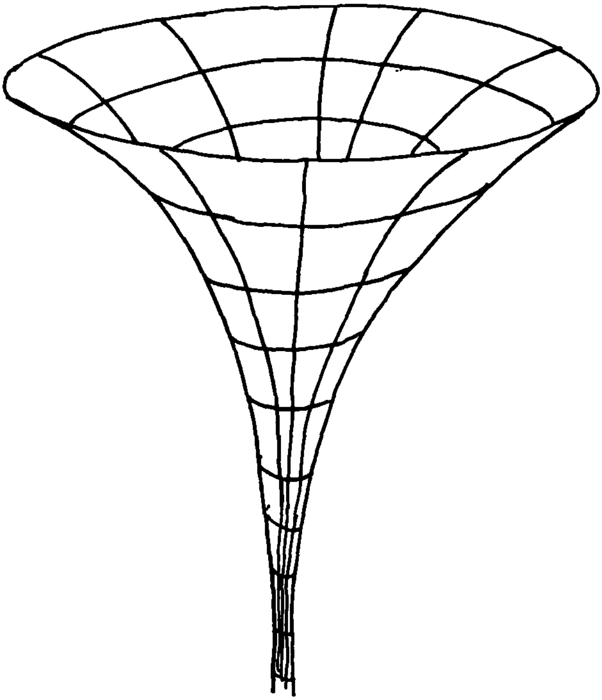

Fig. 3. An expanding constant curvature geometry.

In the s gauge we find only one (2) solutions of the equations of motion for $\lambda > 0$ ($\lambda < 0$). All other solutions are related to those by spatial reparametrization. This is in contrast to the apparently infinite number of solutions in the c gauge. This infinity is illusory because different solutions are related by general coordinate transformations.

4. QUANTIZATION

Consider the system defined by the Hamiltonian density

$$\mathscr{H}(x) = \frac{e\Pi_e^2}{2\beta} + \lambda\beta e \tag{36}$$

and subsidiary conditions

$$P(x)|\psi\rangle = 0 \tag{37a}$$

$$\mathscr{H}(x)|\psi\rangle = 0 \tag{37b}$$

(We assume space is closed so that $0 \leqslant x < 1$.)

We shall construct a quantum system with Hamiltonian (36) that satisfies (37) on a suitable subspace.

The variables $e(x)$ are the absolute values of $g_{11}^{1/2}$ in the S gauge and thus range from 0 to ∞. The variable $\phi = \log e(x)$ ranges from $-\infty$ to $+\infty$ and is an appropriate choice for a canonical variable. The canonical measure

$$\prod_x d\phi(x) \tag{38}$$

is reparametrization invariant unlike $\Pi\,de(x)$. The Lagrangian in terms of ϕ is

$$\beta\left(\dot{\phi}^2\frac{e^{\phi}}{2} + \lambda e^{\phi}\right) \tag{39}$$

so that

$$H = \frac{\Pi_\varphi e^{-\varphi}\Pi_\varphi}{2\beta} + \beta\lambda e^{\phi} \tag{40}$$

where Π_φ is the momentum conjugate to φ. This is the unique ordering which is both Hermitian and reparametrization invariant.

The W–D equation is

$$\left\{-\frac{1}{2\beta}\frac{\delta}{\delta\varphi}e^{-\phi}\frac{\delta}{\delta\varphi} + \beta\lambda e^{\phi}\right\}\psi(\phi) = 0 \tag{41}$$

To solve this we change variables to $e = e^{\phi}$

$$\left(\frac{\delta}{\delta e}\right)^2\psi = 2\lambda\beta^2\psi$$

We now impose (37) by requiring ψ to be a function of the spatial volume

$$\psi(v) = \psi\left(\int e\,dx\right) \tag{42}$$

We obtain

$$\frac{d^2\psi}{dv^2} = 2\lambda\beta^2\psi(v) \tag{43}$$

whose solution is

$$\psi(v) = a\exp(2\lambda\beta^2)^{1/2}v + b\exp(2\lambda\beta^2)^{1/2}v \tag{44}$$

In computing inner products between gauge-invariant states the measure $d\phi$ is replaced by

$$dv/v \tag{45}$$

With this measure the wave function (44) is nonnormalizable for all a, b, λ. For $\lambda > 0$ and $a = 0$ the divergence in the norm of ψ is a mild logarithmic infinity coming from small volumes. Although no covariant physical regulation scheme has been formulated for quantum gravity we assume reasonable physics can be obtained by cutting off small $v < \varepsilon$ and looking for quantities which are insensitive to the cutoff. When $\lambda > 0$ the overwhelming bulk of the probability distribution is concentrated in volumes of order ε. There is no interesting long-distance physics. This is the quantum remnant of the classical collapse discussed in Section 1.

For $\lambda < 0$ general a, b give rise to wave functions with logarithmic divergences both in the uv and ir. The existence of two linearly independent solutions for ψ is related to the two classical motions (34). To see this we compute the expectation value of $\dot{e}/e$ which is positive (negative) for expanding (contracting) geometries. Using

$$\frac{\dot{e}}{e} = \frac{1}{\beta}\Pi_e$$

and (44) we find

$$\left\langle \frac{\dot{e}}{e} \right\rangle = \frac{a^*a - b^*b}{a^*a + b^*b}|2\lambda|^{1/2} \tag{46}$$

Thus for $b = 0$ ($a = 0$) the geometry expands (contracts).

We can also compute the dispersion in $\dot{e}/e$. For the states with a or b vanishing we find

$$\left\langle \left(\frac{\dot{e}}{e}\right)^2 \right\rangle - \left\langle \frac{\dot{e}}{e} \right\rangle^2 = 0 \tag{47}$$

The expansion rate operator whose properties we have just examined describes a geometrical property of the metric and as such is generally coordinate invariant. However, in gauges other than the synchronous gauge it is a nonlocal functional of the metric. We can also study the properties of quantized space-time in terms of more familiar gauge-invariant operators which are integrals of local functions of g. The simplest such operator is the space-time volume

$$\Omega = \int d^2x \sqrt{g}$$

Its expectation value is

$$\langle \Omega \rangle = \frac{T \int_0^\infty \frac{dv}{v} v |\psi|^2}{\int_0^\infty \frac{dv}{v} |\psi|^2}$$

$$= T \begin{cases} 0, & \lambda > 0 \\ \infty, & \lambda \leqslant 0 \end{cases} \tag{48}$$

where T is the proper "lifetime" of the geometry. In evaluating (48) we have introduced an ultraviolet volume cutoff ε. The zero value for $(1/T)\langle\Omega\rangle$ with $\lambda > 0$ is the limit of $-1/\ln \varepsilon$, while the infinity for $\lambda < 0$ is of infrared origin and persists even for finite ε.

Clearly only for $\lambda > 0$ does it make sense to talk of long-distance, cutoff independent properties of space-time.

More interesting than the volume are the curvature invariants:

$$K_n = \int d^2x \sqrt{g} R^n = \int dx\, d\tau \left(\frac{\ddot{e}}{e} \right)^n e \tag{49}$$

The equations of motions give

$$\frac{\ddot{e}}{e} = \frac{\dot{e}}{2e^2} - \lambda \tag{50}$$

so that

$$K_1 = \int dx\, d\tau \left[\frac{\dot{e}^2}{2e} + \lambda e - 2\lambda e \right] = \int d\tau \left(\frac{H}{\beta} - 2\lambda V \right) \tag{51}$$

and

$$\langle K_1 \rangle = -2\lambda \langle \Omega \rangle \tag{52}$$

which agrees with the "classical" statement that the average curvature of space time is -2λ.

For $n \geqslant 2$, K_n suffers from ordering ambiguities. We resolve these by insisting that the classical statement that space-time has constant curvature -2λ remain true at least in the sense of expectation values. The invariant statement of this requirement is

$$\langle K_n \rangle = (-2\lambda)^n \langle \Omega \rangle \tag{53}$$

It may be achieved by the following ordering for K_2:

$$K_2 = \int e \left(\frac{\Pi e^2}{2\beta^2} - \lambda \right)^2$$

and analogous ordering for the higher K_n. This choice of ordering is the only polynomical form for K_n we have found which is Hermitian (with respect to the measure de/e), transforms as a density under spatial coordinate changes, and reproduces equation (53).

The equation $K_1|\psi\rangle = -2\lambda\Omega|\psi\rangle$ is satisfied identically by any state which obeys the Wheeler–DeWitt equation. Thus all fluctuations in the integrated curvature are induced by those in Ω. The average curvature $\Omega^{-1}K_1$ (note $[K_1\Omega] = 0$) behaves classically.

Similarly since $H(x)|\psi\rangle = 0 \Rightarrow (\Pi^2/2\beta^2)(x)|\psi\rangle = -\lambda|\psi\rangle$ we have

$$K_2|\psi\rangle = 4\lambda^2\Omega|\psi\rangle \tag{54}$$

This pattern continues for all n. The general form for K_n is

$$K_n = (-2\lambda)^n \Omega + \sum_{K=0}^{n-1} C_k \mathscr{H} \Pi^{2k} \tag{55}$$

so that

$$K_n|\psi\rangle = (-2\lambda)^n \Omega \; |\psi\rangle \tag{56}$$

and all average curvature invariants are classical.

Operators like the K_n, which are time-reversal invariant, take on the same value for both the expanding ($b = 0$) and contracting ($a = 0$) states since these transform into each other under time reversal.

Operators (like the expansion rate) which are not T invariant are only sharp if we choose a or $b = 0$.

Based on the computations we have done we believe that if such a choice is made all local properties of space–time are classical. All quantum fluctuations are driven by those in the space–time volume.

The wave functions with a or $b = 0$ seem to be the only physically reasonable solutions of the W–D equation. One cannot imagine preparing a state which is a superposition of an expanding and contracting universe. But we are here treading on dangerous ground—the whole question of the meaning of the quantum mechanical measurement theory when applied to the whole universe is a murky area into which we would rather not venture at present.

To summarize, we have formulated pure $(1+1)$-dimensional quantized gravity as a system with one degree of freedom. The quantum mechanics has an ultraviolet divergence problem at small values of the spatial volume v. For positive cosmological constant the whole wave function is concentrated around $v = 0$ and there is no interesting dynamics when the ultraviolet regulator is removed. This reflects the behavior of the classical Lagrangian $\lambda\sqrt{g}$ which predicts a singular collapsed geometry.

For $\lambda < 0$ the expectation values of the spatial and space-time volumes are infinite even when the ultraviolet regulator is removed. The system has two states corresponding to an expanding and collapsing universe. Local geometrical quantities have no quantum fluctuations.

5. COUPLING TO MATTER

To study the coupling of the gravitational field to matter, it is convenient to rewrite the Lagrangian (28) in the axial gauge. The two gauges are related by a time-dependent reparameterization of space which is generally complicated and nonlocal. However, at any fixed time we can set $\partial_x e = 0$ within the synchronous gauge, so that the two gauges coincide at that instant. Since the Lagrangian describes changes in the system over infinitesimal time intervals we need only find the relation between the gauges for configurations which are close to

$$e^{\alpha}_{\mu} = \begin{pmatrix} 1 & 0 \\ 0 & V(\tau) \end{pmatrix} \tag{57}$$

Thus we write

$$e(x,\tau_0)=V(\tau_0)$$
$$e(x,\tau_0+\Delta\tau)=V(\tau_0)+\Delta\tau\,\dot{V}(\tau_0)+\Delta\tau\,\partial x(V(\tau_0)f(x,\tau_0)) \qquad (58)$$

That is we interpret the change in e in the synchronous gauge as the change in e_1^1 ($\equiv V$) in axial gauge plus the transformation between gauges. Under the gauge transformation by $\Delta\tau f(x,\tau_0)$ the other components of e_μ^α transform as

$$\delta e_0^0=0=\delta e_1^0$$
$$\delta e_0^1=-V(\tau_0)f(x,\tau_0)\equiv y(x,\tau_0) \qquad (59)$$

We conclude that

$$\dot{e}(x,\tau_0)=(\dot{V}-\partial xY) \qquad (60)$$

So the Lagrangian density is

$$\mathscr{L}=\frac{\beta}{2V}(\dot{V}-\partial_xY)^2-\beta\lambda V \qquad (61)$$

We now add the coupling to a gauge field

$$\mathscr{L}_{\text{gauge}}=-\frac{1}{4}\sqrt{g}\,g^{\mu\lambda}g^{\nu k}F_{\mu\nu}F_{\lambda k}$$
$$=\frac{1}{2\sqrt{g}}(\dot{A}_1-\partial_xA_0)^2 \qquad (62)$$

In the axial gauge $\sqrt{g}=v$.

In $1+1$ dimensions a gauge field on a closed ring has a single degree of freedom, the Wilson loop around the ring. This degree of freedom is exposed in the Coulomb gauge

$$\partial_xA_1=0$$
$$A_1\equiv A(t) \qquad (63)$$

The total Lagrange density is then

$$\mathscr{L}=\frac{\beta}{2v}\left[(\dot{v}-\partial_xY)^2+(\dot{A}-\partial_xA_0)^2\right]-\beta\lambda V \qquad (64)$$

and the Hamiltonian density

$$\mathscr{H}(x) = \frac{v}{2\beta}(P^2 + E^2) + \beta\lambda v$$

$$= [P\partial_x Y + E\partial_x A_0] - \frac{1}{2v}\left[(\partial_x Y)^2 + (\partial_x A_0)^2\right] \tag{65}$$

where p and E are the canonical conjugates to V and A, respectively.

The constraint equations for y and A_0, combined with periodicity in x, imply that

$$y = A_0 = 0 \tag{66}$$

so that

$$\mathscr{H}(x) = \frac{v}{2\beta}(P^2 + E^2) + \lambda v \tag{67}$$

The electric field E clearly commutes with $H(x)$ so we can find solutions of the Wheeler–DeWitt equation

$$\mathscr{H}(x)|\psi\rangle = 0 \tag{68}$$

by first diagonalizing E and then finding solutions of

$$\left[\frac{v}{2\beta}P^2 + \left(\lambda + \frac{E^2}{2}\right)v\right]\psi_E(v) = 0 \tag{69}$$

Clearly the presence of a nonzero field E shifts the cosmological constant just as a θ parameter does in four dimensions. The coupled system has an infinite number of states. If the original cosmological constant is negative some of these states witll represent expanding finite worlds but there is a critical value of E above which $\langle\Omega\rangle = 0$.

The introduction of scalar or spinor fields is more complicated and we have not yet found a consistent quantum mechanical description of these systems. We will therefore treat these fields classically. However, there is one quantum mechanical effect of scalars which can (and for consistency should) be included in our semiclassical treatment. Polyakov (1981) has shown that the short-distance quantum fluctuations of scalar fields produce a renormalization of the effective gravitational action. The cosmological constant is infinitely renormalized and $\beta \to \beta - n/24\pi$, where n is the number of scalar fields.

We will assume that it is meaningful to separate these short distance effects from the rest of the dynamics of the field, which we will treat classically.

It is relatively easy to extend our semiclassical considerations to include scalar fields. We will restrict our attention to "minimal" couplings, for which derivatives of $g_{\mu\nu}$ do not appear in the matter Lagrangian. The semiclassical prescription is then to solve the Euler–Lagrange equations of

$$\beta\left(\frac{\dot{e}^2}{2e} - \lambda e\right) + \mathscr{L}_{\text{matter}} = \mathscr{L} \tag{70}$$

Together with the constraints ($\pi_e = \dot{e}/e$ because of minimal coupling)

$$\frac{e\Pi_e^2}{2\beta} + \lambda\beta e + \mathscr{H}_m(x) = 0 = \mathscr{H}(x) \tag{71}$$

$$P = e\partial_x\Pi_e + P_m = 0 \tag{72}$$

$\mathscr{H}_m$ and P_m are the energy and momentum densities of the matter fields.

We must make sure that the constraint (71) propagates [(72) generates a symmetry of $\mathscr{L}$and is certainly conserved.] This entails

$$\dot{\mathscr{H}}(y) = \int dx\,\{\mathscr{H}(x), \mathscr{H}(y)\}_{\text{PB}} = 0 \tag{73}$$

$$\int \{\mathscr{H}_m(x), \mathscr{H}_m(y)\}_{\text{PB}}\,dx = 0 \tag{74}$$

In (74) we have used the minimality constraint.

The classical matter energy densities satisfy the Dirac Schwinger, PB algebra so (74) can be rewritten

$$0 = \partial_x\left[\frac{1}{e^2(x)}P_m(x)\right] \tag{75}$$

which, in view of (72) is equivalent to

$$0 = \partial_x\left[\frac{1}{e(x)}\partial_x\Pi_e\right] \tag{76}$$

At any instant of time we can choose a parametrization for which $\partial_x e = 0$. (76) then implies

$$\partial_x^2\Pi_e = 0 \tag{77}$$

whose only periodic solution is $\Pi_e = \text{const}$. Since Π_e transforms like a scalar under reparametrizations $\Pi_e = \text{const}$ even if $\partial_x e \neq 0$. Thus Π_e is constant in x for all times and if we choose $\partial_x e = 0$ initially the geometry will remain homogeneous. Thus it is apparently impossible to produce spatially varying geometries in $1+1$ dimensions, even in the presence of matter.

Returning to equation (72) we find a constraint on the matter fields

$$P_m(x) = 0 \tag{78}$$

for a single massless scalar field (78) can be satisfied if only the zero momentum mode of the field is excited. For several fields there are many solutions of the field equations satisfying (78).

We are not satisfied with our understanding of the quantum mechanical implementation of all these constraints. One possible procedure is to solve the constraints classically and then quantize the remaining modes. For a single scalar field this means that we quantize only the zero momentum mode.

The coupling to matter is particularly simple when the trace of the stress energy tensor vanishes, as it does for a free massless field:

$$\mathscr{L}_m = \frac{1}{2}\sqrt{g}\, g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi \tag{79}$$

The Hamiltonian density is then

$$\mathscr{H}_m = \frac{1}{2e}\left[\Pi_\varphi^2 + (\partial_x\varphi)^2\right] \tag{80}$$

and the equations of motion are

$$\frac{\ddot{e}}{e} = \frac{1}{e}\left[\frac{1}{2}\frac{\dot{e}^2}{e} - \lambda e + e\frac{\dot{\varphi}^2}{2} + \left(\frac{\nabla\phi}{2}\right)^2\right] \tag{81}$$

$$\mathscr{H}(x) = \left[\frac{1}{2}\frac{\dot{e}^2}{e} + \lambda e + e\frac{\dot{\varphi}^2}{2} + \frac{(\nabla\phi)}{2}\right] = 0 \tag{82}$$

Equations (81) and (82) can be rewritten

$$\ddot{e}/e = -2\lambda \tag{83}$$

$$\frac{1}{2}\dot{e}^2/e + \lambda e + \frac{P^2}{2e} = 0 \tag{84}$$

where P is the canonical conjugate to ϕ and we have used homogeneity to set $\nabla\phi = 0$. The equations of motion for ϕ imply that P is const.

From (83) and (84) we find

$$e = A\cosh(2|\lambda|)^{1/2}t$$

$$A^2 = -\frac{P^2}{2\lambda}$$

This describes a universe which undergoes a cosmological bounce. It starts at infinite volume, contracts to circumference A at $t = 0$, and reexpands.

6. CONCLUSIONS

We believe we have presented a consistent quantization of (1 + 1)-dimensional gravity in the synchronous gauge. In many respects this gauge is similar to the temporal ($A_0 = 0$) gauge in (1 + 1)-dimensional gauge theories. The value of such a quantization is that it exposes most clearly the physical degrees of freedom. Gauge fields may also be included. Their physical effect is to introduce a "background" electric field which shifts the cosmological constant. Scalar and spinor fields are more difficult and we do not entirely understand how to satisfy the various constraints involved.

One peculiar feature of the (1 + 1)-dimensional theory is that the geometry is completely homogeneous with respect to spatial translations, even in the presence of matter fields. This is not a statement about expectation values, but rather an exact feature of the quantum wave function. For example, the space derivatives of the space-time curvature and the expansion rate $\dot{e}/e$ annihilate the physical states.

The Wheeler–Dewitt equation $\mathscr{H}(x) = 0$ plays a dual role in our formalism. First of all it is the condition for invariance under local time translations. It also seems to play a gauge fixing role, at least classically, which picks out a particular family of s gauges. In fact $\mathscr{H}(x) = 0$ fixes the gauge freedom associated with local time translations of the initial spacelike surface Γ. One may then wonder whether the theory is invariant under these transformations. Let us consider such a coordinate transformation

$$\delta x = f'(x, \tau)$$
$$\delta\tau = f^0(x) \tag{85}$$

with

$$\dot{f}_1 = -\frac{1}{e^2}\partial_1 f^0 \tag{86}$$

The change in the action may readily be computed.

$$\int \frac{1}{e}\left(\partial_x \frac{\dot{e}}{e}\right)\partial_x f^0 \tag{87}$$

$$= \int \frac{1}{e^2} P(x)\,\partial_x f^0 \tag{88}$$

The important thing about (88) is that it is proportional to $P(x)$ which annihilates all physical states. Therefore, in the physical subspace the theory is invariant under the entire residual gauge group which preserves the form of the s gauge.

On the basis of dimensional considerations, quantum gravity in $1+1$ dimensions should be renormalizable but not necessarily finite. Indeed we find the wave function $\psi(v)$ is not quite normalizable. Its norm is logarithmically ultraviolet divergent. However, many quantities have finite limits as the cutoff length tends to zero. In particular the space-time curvature and local expansion rate are finite.

The behavior of the solution is quite different for $\lambda > 0$ and $\lambda \leqslant 0$. For $\lambda > 0$ the geometry collapses. For $\lambda \leqslant 0$, however, the wave functions describe exponentially expanding or contracting universes. When matter fields are included it seems likely that the expanding or contracting solutions are replaced by a single "bounce" in which an initially contracting universe reaches some minimal size and then expands.

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